Separable quantum states do not have stronger correlations than local realism. A comment on quant-ph/0611126 of Z. Chen.

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Chen (quant-ph/0611126) has recently claimed "exponential violation of local realism by separable states", in the sense that multi-partite separable quantum states are supposed to give rise to correlations and fluctuations that violate a Bell-type inequality that Chen takes to be satisfied by local realism. However, this can not be true since all predictions (including all correlations and fluctuations) that separable quantum states give rise to have a local realistic description and thus satisfy all Bell-type inequalities, and this holds for all number of parties. Since Chen claims otherwise by presenting a new inequality, claimed to be a Bell-type one, which separable states supposedly can violate, there must be a flaw in the argumentation. I will expose this flaw, not merely for clarification of this issue, but perhaps even more importantly since it re-teaches us an old lesson John Bell taught us over 40 years ago. I will argue that this lesson provides us with a new morale especially relevant to modern research in Bell-type inequalities.

Introduction

Chen [1] has recently claimed "exponential violation of local realism by separable states", in the sense that multi-partite separable quantum states are supposed to give rise to correlations and fluctuations that violate a Bell-type inequality that Chen takes to be satisfied by local realism. The violation is claimed to be by an exponentially increasing amount as the number of particles n grows ($n > 2$). In fact, it is claimed to violate the local realistic maximum by a factor of 2^{n-2}.

However, this can not be true since all predictions (including all correlations and fluctuations) that separable quantum states give rise to have a local realistic description and thus satisfy all Bell-type inequalities, and this holds for all number of parties. Although this is known already, let me nevertheless show this first (for the sake of completeness of the discussion) after which I will continue my comment on the work of Chen. Here, I will generalise the exposition by Żukowski [2] to the n-party case.

Any n-party Bell-type inequality has the following generic form

$$|\sum_{k_1,k_2,\dots,k_n} c(k_1,k_2,\dots,k_n)E(k_1,k_2,\dots,k_n)| \leq B(c), \tag{1}$$

where k_1, k_2, etc. are labels that distinghuish various (discrete or continuous) measurement settings that can be measured on system 1, 2, etc., $c(k_1,k_2,\dots,k_n)$ are certain constant coefficients, $B(c)$ is the maximum value obtainable by a local realistic theory for the expression on the left hand side of Eq. (1), and $E(k_1,k_2,\dots,k_n)$ is the 'correlation function' for outcomes of measurement with settings $k_1,k_2,\dots,k_n$, which for local realistic theories is assumed to be the expectation of the product of the local observables:

$$E(k_1,k_2,\dots,k_n) = \int_\Lambda A_1(k_1,\lambda)A_2(k_2,\lambda)\dots A_n(k_n,\lambda)\,p(\lambda)\,d\lambda. \tag{2}$$

Here λ is an integration variable, often denoted as the hidden variable, the set Λ is the set of hidden variables, and the functionals $A_i(k_i,\lambda)$ are either the outcomes ($+1$ or -1) of the measurements determined by the settings k_i and hidden variable λ in case we are dealing with a deterministic theory, or they are the expectation values (in the interval $[-1, 1]$) of these outcomes in the case of a stochastic (non-deterministic) theory, and finally $p(\lambda)$ is a normalised probability distribution of the variable λ. The Bell inequalities of Eq. (1) follow solely from the assumption of Eq. (2).

Let us now suppose that we have an n-partite fully separable quantum state ρ, i.e., a convex sum of product states: $\rho = \sum_j p_j\rho_j^1\otimes\rho_j^2\otimes\dots\otimes\rho_j^n$, with ρ_j^i a density operator for party (subsystem) i. The index j is a summation or integration variable and $\sum_j p_j = 1$, $0 \leq p_j \leq 1$. The correlations these states give rise to all have the form

$$\begin{aligned}&\langle\hat A_1(k_1)\hat A_2(k_2)\hat A_n(k_n)\rangle_\rho := \\ &\mathrm{Tr}[\rho\,\hat A_1(k_1)\otimes\hat A_2(k_2)\dots\otimes\hat A_n(k_n)] = \\ &\sum_j p_j\,\mathrm{Tr}^1[\rho_j^1\hat A_1(k_1)]\mathrm{Tr}^2[\rho_j^2\hat A_2(k_2)]\dots\mathrm{Tr}^n[\rho_j^n\hat A_n(k_n)],\end{aligned} \tag{3}$$

with $\mathrm{Tr}^i[\,\cdot\,]$ the partial trace for party i (i.e., the rest of the parties is traced out), and $\hat A_i(k_i)$ the operator associated with the measurement on party i with setting k_i. Each such measurement has possible outcomes $+1$ or -1, just as was the case above [3]. Since we have that $|\mathrm{Tr}^i[\rho^i\hat A_i(k_i)]| = |\langle\hat A_i(k_i)\rangle_\rho| \leq \max A_i(k_i,\lambda)$ the correlations of Eq. (3) can be written as in Eq. (2) and thus they can be reproduced by a local realistic theory. They thus must satisfy all Bell-type inequalities of Eq. (1), i.e., 'those known at present, as well as those which one day would be derived'[2]. Note that the same holds for all fluctuations a separable state can give rise to, cf. [4].

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Since Chen claims otherwise by presenting a new inequality, claimed to be a Bell-type one, which separable states supposedly can violate, there must be a flaw in the argumentation. I will expose this flaw, not merely for clarification of the issue, but perhaps even more importantly since it re-teaches us an old lesson John Bell has made over 40 years ago, although in a different form. I will argue this lesson provides us with a new morale especially relevant to modern research in Bell inequalities.

Review of Chen's results

Let me first present the result of Chen's analysis. He considers the well-known Mermin-Roy-Singh-Ardehali-Belinskiĭ-Klyshko inequality for n-parties [5]. This inequality is characterised by a specific choise [6] of coefficients $c(k_1, k_2, \ldots, k_n)$ in Eq. (1), where each party chooses between two observables (i.e., each k_i has two possibilities) that are furthermore dichotomic (± 1 valued). Chen takes the inequality to be normalised so that $B(c) = 1$, and chooses a specific choice of measurement settings $k_1, k_2 \ldots, k_n$, such that the two possible local settings for each party are given by orthogonal vectors (i.e., $A_i(k_i) \perp A_i(k_i'))$. Suppose that for this specific choice of settings we define the so-called Bell polynomial [7]

$$\mathcal{M}_n(\lambda) := \sum_{k_1,k_2,\ldots,k_n} c(k_1, k_2, \ldots, k_n) A_1(k_1, \lambda) A_2(k_2, \lambda) \ldots A_n(k_n, \lambda). \tag{4}$$

It follows from $A_i(k_i, \lambda) = \pm 1$ for all i that for the specific settings used $-1 \leq \mathcal{M}_n(\lambda) \leq 1$. Because of linearity of the mean (i.e., the linear combination in Eq. (1) can also be evaluated under the expectation value) the Bell inequality Eq. (1) (for the specific choise of coefficients $c(k_1, k_2, \ldots, k_n)$) can be reformulated as an upperbound on the expectation of this Bell polynomial $\mathcal{M}_n(\lambda)$. Indeed, local realism predicts the result that

$$|\langle \mathcal{M}_n \rangle_{\mathrm{LHV}}| = |\int \mathcal{M}_n(\lambda)\, p(\lambda)\, d\lambda| \leq 1, \tag{5}$$

with the expectation value $\langle X \rangle_{\mathrm{LHV}} := \int_\lambda X(\lambda) p(\lambda)\, d\lambda$. This is in fact a shorthand notation for the Bell-type inequality

$$|\sum_{k_1,k_2,\ldots,k_n} c(k_1, k_2, \ldots, k_n) E(k_1, k_2, \ldots, k_n)| \leq 1, \tag{6}$$

with the correlation $E(\,\cdot\,)$ given by Eq. (2). This the Mermin-Roy-Singh-Ardehali-Belinskiĭ-Klyshko inequality [5].

The quantum mechanical counterpart of this inequality is obtained as follows. Choose $p(\lambda) = \delta(\lambda - \rho)$ with ρ a quantum mechanical state of n qubits, and next substitute the measurement functionals in the Bell polynomial by the operators associated to the measurements to give its quantum counterpart which is denoted by the operator $\hat{\mathcal{M}}_n$:

$$\hat{\mathcal{M}}_n := \sum_{k_1,k_2,\ldots,k_n} c(k_1, k_2, \ldots, k_n) \hat{A}_1(k_1) \hat{A}_2(k_2) \ldots \hat{A}_n(k_n). \tag{7}$$

The local settings are anticommuting (i.e. $\{\hat{A}_i(k_i), \hat{A}_i(k_i')\} = 0$) since this incorporates the local orthogonality of the dichotomic observables.

After this conversion we obtain the well-known result [5] that quantum mechanics obeys the inequality

$$|\langle \hat{\mathcal{M}}_n \rangle_\rho| \leq 2^{(n-1)/2}, \tag{8}$$

where the expectation value $\langle \hat{X} \rangle_\rho := \mathrm{Tr}[\hat{X}\rho]$. The upperbound can be achieved by the maximally entangled GHZ states.

If we set $n = 2$ in Eq. (6) we obtain the original CHSH inequality for local realism and for $n = 2$ in Eq. (8) we get the Tsirelson inequality for quantum mechanics.

Chen now considers the quantum mechanical operator

$$\hat{\mathcal{V}}_n := \hat{\mathcal{M}}_n + \hat{\mathcal{M}}_n^2, \tag{9}$$

and shows that

$$\begin{aligned} \langle \hat{\mathcal{V}}_n \rangle_\rho &:= \mathrm{Tr}[\rho \hat{\mathcal{V}}_n] \\ &= \mathrm{Tr}[\rho \hat{\mathcal{M}}_n] + \mathrm{Tr}[\rho \hat{\mathcal{M}}_n^2] \\ &= \langle \hat{\mathcal{M}}_n \rangle_\rho + \langle \hat{\mathcal{M}}_n \rangle_\rho^2 + \Delta(\hat{\mathcal{M}}_n)_\rho, \end{aligned} \tag{10}$$

with $\Delta(\hat{\mathcal{M}}_n) = \langle (\hat{\mathcal{M}}_n - \langle \hat{\mathcal{M}}_n \rangle_\rho)^2 \rangle_\rho$ the variance of $\hat{\mathcal{M}}_n$ in the state ρ. For separable quantum states ρ_{sep} he obtains the bound $\langle \hat{\mathcal{V}}_n \rangle_{\rho_{\mathrm{sep}}} \leq 2^{n-1}$, which is tight since it can be achieved by a separable state (see [8]).

Local realism gives Eq. (5), and Chen furthermore claims that

$$\begin{aligned} \Delta(\mathcal{M}_n)_{\mathrm{LHV}} &:= \int (\mathcal{M}_n(\lambda) - \langle \mathcal{M}_n \rangle_{\mathrm{LHV}})^2\, p(\lambda)\, d\lambda \\ &= \langle \mathcal{M}_n^2 \rangle_{\mathrm{LHV}} - \langle \mathcal{M}_n \rangle_{\mathrm{LHV}}^2 && (11) \\ &\leq 1 - \langle \mathcal{M}_n \rangle_{\mathrm{LHV}}^2, && (12) \end{aligned}$$

from which it follows that

$$\begin{aligned} \langle \mathcal{V}_n \rangle_{\mathrm{LHV}} &:= \langle \mathcal{M}_n \rangle_{\mathrm{LHV}} + \langle \mathcal{M}_n^2 \rangle_{\mathrm{LHV}} \\ &= \langle \mathcal{M}_n \rangle_{\mathrm{LHV}} + \langle \mathcal{M}_n \rangle_{\mathrm{LHV}}^2 + \Delta(\mathcal{M}_n)_{\mathrm{LHV}} \leq 2. \end{aligned} \tag{13}$$

Chen thus obtains that $\langle \mathcal{V}_n \rangle_{\mathrm{LHV}} \leq 2$ whereas $\langle \hat{\mathcal{V}}_n \rangle_{\rho_{\mathrm{sep}}} \leq 2^{n-1}$ and since this last bound can be achieved by separable states, he concludes that for $n > 2$ they violate the local realistic inequality by an exponentially large factor. This conclusion contradicts the previous analysis that the correlations in separable states can be reproduced by local realism. So where did Chen's analysis go wrong?

Exposing the problematic relation

Let us first note that in Eq. (12) it must have been used that

$$\langle \mathcal{M}_n^2 \rangle_{\mathrm{LHV}} = \int (\mathcal{M}_n(\lambda))^2\, p(\lambda)\, d\lambda \leq 1, \tag{14}$$

which follows from $(\mathcal{M}_n(\lambda))^2 \leq 1$ that on its turn follows from the fact that $-1 \leq \mathcal{M}_n(\lambda) \leq 1$.

Using this we see that Chen's claim follows solely from the quantum mechanical inequality

$$\langle \hat{\mathcal{M}}_n^2 \rangle_\rho \leq 2^{n-1}, \tag{15}$$

and the fact that Eq. (15) can be saturated for separable states. Comparing this to Eq. (14) it would seem that already for $n = 2$ separable states give rise to correlations that can not be reproduced by local realism.

However, it is of course not $\mathcal{M}_n(\lambda)$ that is measured in any experiment, but the observables $A_i(k_i, \lambda)$. But this captures only part of the problem since, as we have seen $\mathcal{M}_n(\lambda)$ can nevertheless be used to obtain a legitimite Bell-type inequality (i.e. Eq. (6)), whereas $(\mathcal{M}_n(\lambda))^2$ can not – or so I claim. What accounts for this difference?

A first starting point is to note that the operator $\hat{\mathcal{V}}_n = \hat{\mathcal{M}}_n + \hat{\mathcal{M}}_n^2$ of Eq. (9) should be translated into its hidden variable counterpart as $\mathcal{V}_n(\lambda) = \mathcal{M}_n(\lambda) + \mathcal{M}_n^2(\lambda)$. But in order for Chen's analysis to go through, he must have translated this into $\mathcal{V}_n(\lambda) = \mathcal{M}_n(\lambda) + (\mathcal{M}_n(\lambda))^2$. Chen must thus have assumed that $\mathcal{M}_n^2(\lambda) = (\mathcal{M}_n(\lambda))^2$, and since $\hat{\mathcal{M}}_n$ and $\hat{\mathcal{M}}_n^2$ commute this seems a reasonable requirement. Indeed, but only if the quantity $\mathcal{M}_n(\lambda)$ can be considered unproblematically as a hidden variable observable. However, I will now argue that this is not the case.

Let's take a closer look at the Bell polynomial $\mathcal{M}_n(\lambda)$ of Eq. (4), and — one is reminded of Bell's 1966 critique [9] on von Neumann's 'no-go theorem' — it then becomes apparent that the definition of $\mathcal{M}_n(\lambda)$ uses a suspicious additivity of incompatible observables, since it involves different local setting ($k_i \neq k_i'$). Because of this the Bell polynomial of Eq. (4) can not be considered to be an observable that local realism determines, and is never measured as such, since, as Bell has taught us, "a measurement of a sum of noncommuting observables cannot be made by combining trivially the results of separate observations on the two terms – it requires a quite distinct experiment. [...] But this explanation of the nonadditivity of allowed values also established the nontriviality of the additivity of expectation values. The latter is quite a peculiar property of quantum mechanical states, not to be expected *a priori*. There is no reason to demand it individually of the hypothetical dispersion free states [hidden variable states λ], whose function it is to reproduce the *measurable* peculiarities of quantum mechanics when *averaged over.* "[9]. For Bell an expression involving noncommuting spin observables such as $[\sigma_x + \sigma_y](\lambda)$ could not be assumed to be equal to $\sigma_x(\lambda) + \sigma_y(\lambda)$.

If we apply Bell's lesson to Chen's analysis we obtain that measurement of the Bell polynomial $\mathcal{M}_n(\lambda)$ can not be made by combining trivially the results of noncommuting observables. The hidden variables λ only determine the outcomes $A_i(k_i, \lambda)$ of individual measurements (with settings k_i) and not the outcomes of measurement of the quantity $\mathcal{M}_n(\lambda)$ since the latter involves incompatible observables. The only function of the Bell polynomial $\mathcal{M}_n(\lambda)$ is to allow for a shorthand notation of the Bell-type inequalities. Indeed, when averaged over λ it gives the inequality Eq. (5), which by using linearity of the mean can be rewritten as a sum of expectation values in a legitimate local realistic form, namely as the legitimate Bell-type inequality of Eq. (6) that local realism must satisfy. Indeed, all expectation values in the Bell-type inequality Eq. (6) involve only commuting (compatible) quantities, and no noncommuting ones. So although $\mathcal{M}_n(\lambda)$ cannot be thought of as an observable specifying the sum of local realistic quantities determined by the hidden variable λ, and as such is never measured in an experiment, when averaged over it does allow for a legitimate shorthand notation of the Bell-type inequality Eq. (6). This is the reason why the hidden variable counterpart of the operator $\hat{\mathcal{M}}_n$ can be safely chosen to be the Bell-polynomial $\mathcal{M}_n(\lambda)$.

However, I will now argue that from a local realistic point of view such a manouvre cannot performed for the functional $(\mathcal{M}_n(\lambda))^2$.

Firstly, $(\mathcal{M}_n(\lambda))^2$ is not the legitimite hidden variable counterpart of the operator $(\hat{\mathcal{M}}_n)^2$. Thus when averaged over it gives the inequality Eq. (14) but this is not a legitimite local realistic counterpart of the inequality Eq. (15). This can be easily seen from the following example. Suppose we take $n = 2$ and use the observables A and A' and B and B' for party 1 and 2 respectively. Then if we expand $(\mathcal{M}_2(\lambda))^2$ we get

$$\begin{aligned}(\mathcal{M}_2(\lambda))^2 \;=\; & \frac{1}{4}(A^2 + A'^2)(B^2 + B'^2) + \frac{1}{2}AA'(B^2 - B'^2) \\ & + \frac{1}{2}BB'(A^2 - A'^2),\end{aligned} \tag{16}$$

where the dependency of A, A', B, and B' on λ has been omitted for clarity. However, if we expand $(\hat{\mathcal{M}}_2)^2$ (and using the local anticommutativity) we get

$$\begin{aligned}(\hat{\mathcal{M}}_2)^2 &= \frac{1}{4}(\hat{A}^2 + \hat{A}'^2) \otimes (\hat{B}^2 + \hat{B}'^2) - (\hat{A}\hat{A}' \otimes \hat{B}\hat{B}') \\ &= \frac{1}{4}(\hat{A}^2 + \hat{A}'^2) \otimes (\hat{B}^2 + \hat{B}'^2) + (\hat{A}'' \otimes \hat{B}''),\end{aligned} \tag{17}$$

with $\hat{A}'' = [\hat{A}, \hat{A}']/2i$ and $\hat{B}'' = [\hat{B}, \hat{B}']/2i$ some self adjoint operators (with eigenvalues ± 1) that can be thought to correspond to some well defined observables. We indeed see that Eq. (17) is structurally different from the local realistic expression Eq. (16), and they can therefore not be considered to be the counterpart of eachother. The correct local realistic counterpart of $(\hat{\mathcal{M}}_2)^2$ is obtained by translating Eq. (17) directly into

$$\mathfrak{M}_2(\lambda) := \frac{1}{4}(A^2 + A'^2)(B^2 + B'^2) + A''B'', \tag{18}$$

with A'' and B'' some dichotomic ± 1 valued observables that are the local realistic counterpart of respectively $\hat{A}''$ and $\hat{B}''$. (The dependency of the right hand side quantities on λ has again been omitted for clarity). The functional $\mathfrak{M}_2(\lambda)$ (and *not* $(\mathcal{M}_2(\lambda))^2$) is the Bell-polynomial that, when averaged over and using linearity of the mean, gives the Bell-type inequality which is the counterpart of the quantum mechanical inequality using $(\hat{\mathcal{M}}_2)^2$.

Secondly, when averaged over $(\mathcal{M}_n(\lambda))^2$ does not give a shorthand notation for a Bell-type inequality or any other constraint which local realism must obey. Invoking linearity of the mean does not help here.

The reason for this is that we end up with the above mentioned problem that Bell pointed out: we get expectation values of the products of noncommuting (incompatible) observables which cannot be determined by combining measurements of the individual observables. To see this we expand $\mathcal{M}_n(\lambda))^2$ in $\langle(\mathcal{M}_n(\lambda))^2\rangle_{\mathrm{LHV}}$, as was done in Eq. (16) for $n = 2$. We then get terms like $\langle[\ldots A_i(k_i,\lambda)A_i(k_i',\lambda)\ldots]\rangle_{\mathrm{LHV}}$, which for $k_i \neq k_i'$ involve incompatible experiments that correspond to locally anticommuting operators in quantum mechanics. Indeed, it is precisely this local noncommutativity, i.e., $\hat{A}_i(k_i)\hat{A}_i(k_i') = -\hat{A}_i(k_i')\hat{A}_i(k_i)$, that has no counterpart for local realistic observables, which is responsible for the structural differences in Eq. (16) and Eq. (17) and which Chen uses to get the exponentially diverging result that $\langle(\mathcal{M}_n(\lambda))^2\rangle_{\mathrm{LHV}} \leq 1$ whereas $\langle\hat{\mathcal{M}}_n^2\rangle_\rho \leq 2^{(n-1}$ (with the latter tight for separable quantum states). But we have seen that for local realism these expectation values of the products of noncommuting (incompatible) observables are problematic for precise the same reason as why additivity of expectation values is problematic for the sum of noncommuting observables: measurement of the product of noncommuting observables requires a quite distinct experiment from the experiments used to measure the individual terms in the product.

Thus Bell's critique on von Neumann's no-go theorem equally well applies here too: "[...] the formal proof does not justify his informal conclusion"[9], i.e., although Chen's proof is mathematically correct and as such is interesting, his conclusion is nevertheless wanting since the local hidden variable theorist would not be enforced to regard the assumption Eq. (14) as the legitimite counterpart of Eq. (15), nor to regard $(\mathcal{M}_n(\lambda))^2$ as a legitimite shorthand notation for any sort of Bell-type inequality. Chen's analysis thus breaks down.

It is thus not the strength of correlations or fluctuations in separable states which ruled out local realism, but "[i]t was the arbitrary assumption of a particular (and impossible) relation between he results of incompatible measurements either of which *might* be made on a given occasion but only one of which can in fact be made."[9]

Repercussions for modern research on Bell-type inequalities

In modern research on Bell-type inequalities (see [1, 5, 7, 10], however cf. [11]) one often considers recursive definitions and shorthand notations in terms of Bell polynomials [7] (e.g., see Eq. (4)) and their quantum mechanical counterparts, the so called Bell operators. The latter are particular linear combinations of operators that correspond to products of local observables. Examples of such Bell-operators are $\hat{\mathcal{M}}_n$ in the multi-partite setting or for example $\hat{\mathcal{B}} = \hat{A}\hat{B} + \hat{A}\hat{B}' + \hat{A}'\hat{B} - \hat{A}'\hat{B}'$ for the bi-partite setting. In quantum mechanics these operators $\hat{\mathcal{M}}_n$ and $\hat{\mathcal{B}}$ can be considered to be observables themselves since a sum of self-adjoint operators is again self-adjoint and every self-adjoint operator is supposed to correspond to an observable. Furthermore, the additivity of operators gives additivity of expectation values. (This is the reason why the operators $\hat{\mathcal{M}}_n$ and $\hat{\mathcal{M}}_n^2$ that Chen uses can be considered to be proper quantum mechanical observables.)

Thus the so called Tsirelson inequality expressed as

$$|\langle\hat{A}\hat{B}\rangle_\rho + \langle\hat{A}\hat{B}'\rangle_\rho + \langle\hat{A}'\hat{B}\rangle_\rho - \langle\hat{A}'\hat{B}'\rangle_\rho| \leq 2\sqrt{2} \tag{19}$$

can thus equally well be expressed in a shorthand notation as $|\langle\hat{\mathcal{B}}\rangle_\rho| \leq 2\sqrt{2}$, with $\hat{\mathcal{B}} = \hat{A}\hat{B} + \hat{A}\hat{B}' + \hat{A}'\hat{B} - \hat{A}'\hat{B}'$.

However, as noted by Bell and as cited before in this note, this is additivity of expectation values is "a quite peculiar property of QM, not to be expected *a priori*" for the hidden variable states λ. Thus for Bell-type inequalities such a shorthand notation involves taking care of some crucial subtleties. I will now discuss three such subtleties that should be taken into account in deriving Bell-type inequalities, and then relate them to the discussion of the previous section.

(I) Firstly, the local realistic Bell polynomials are not to be regarded as observables. The danger is that because the operator identity in quantum mechanics $\hat{\mathcal{B}} = \hat{A}\hat{B} + \hat{A}\hat{B}' + \hat{A}'\hat{B} - \hat{A}'\hat{B}'$ does indeed define a new observable $\hat{\mathcal{B}}$, one is tempted to formulate the hidden variable counterpart as:

$$\mathcal{B}(\lambda) = A(\lambda)B(\lambda) + A(\lambda)B'(\lambda) + A'(\lambda)B(\lambda) - A'(\lambda)B'(\lambda). \tag{20}$$

However, if $\mathcal{B}(\lambda)$ is not regarded as merely a shorthand notation for the sum of the four terms in Eq. (20), but is supposed to be the counterpart of the observable $\hat{\mathcal{B}}$, Eq. (20) involves the problematic additivity of eigenvalues, which cannot be demanded of local realism. Indeed, four different non-compatible setups are involved and not just one, which the notation $\mathcal{B}(\lambda)$ when considered as a single observable could suggest.

Thus, when deriving local hidden variable observables that depend on the hidden variable λ, only compatible experimental setups must be considered. The difficulty of measuring incompatible observables thus has to be explicitly taken into account in the hidden variable expression. This is different from quantum mechanics where the incompatibility structure already is captured in the (non-)commutativity structure of the operators that correspond to the observables in question.

(II) Secondly, when using a shorthand notation it must be possible (by for example using linearity of the mean) to translate the shorthand notation into a legitimate Bell-type inequality which a local realist cannot but accept or into any other legitimate local realistic constraint. As an example, note that we have seen that $\mathcal{M}_n(\lambda)$ did allow for formulating a legitimite Bell inequality whereas $(\mathcal{M}_n(\lambda))^2$ did not.

(III) Thirdly, suppose one would indeed regard the functionals $\mathcal{M}_n(\lambda)$ and $\mathcal{B}(\lambda)$ to be the quantities of interest and regard them as observables. The first subtlety mentioned above shows that this is unproblematic only if they are thought of as being genuine irreducible observables and not to be composed out of other incompatible observables. Then to be fair to local realism from the start the possible values of measurement of these observables $\mathcal{M}_n(\lambda)$ and $\mathcal{B}(\lambda)$ in the local realist model should then be equal to the eigenvalues of the quantum mechanical counterparts $\hat{\mathcal{M}}_n$ and $\hat{\mathcal{B}}$. And these eigenvalues are

$\{2^{(n-1)/2},\ -2^{(n-1)/2},\ 0\}$ and $\{2\sqrt{2},\ -2\sqrt{2},\ 0\}$ respectively. The possible outcomes for the local realist quantities should equal these eigenvalues, and then no violation can be seen to occur. Indeed, predictions for a single observation can always be mimicked by a local realistic model. Furthermore, if we now go back to the problematic operator $\hat{\mathcal{M}}_n^2$ that Chen considered, we see that it has eigenvalues $\{2^{(n-1)},\ 0\}$, (see footnote [8]), whereas it was assumed that its local realistic counterpart has outcomes in $[-1,1]$, and can thus by construction, and not because of local realism, never reproduce the quantum outcomes.

Using these three subtleties we can now understand in a different way where Chen's analysis has gone astray: (i) If we think of Eq. (14) as a mere shorthand notation of a complex summation, then the second subtlety is the major problem. On this reading the functional $(\mathcal{M}_n(\lambda))^2$ cannot be taken to give a constraint (i.e., Eq. (14)) which is a shorthand notation of a legitimate Bell-type inequality. (ii) However, one can argue that the analysis of Chen does not treat the Bell polynomial $\mathcal{M}_n(\lambda)$ in Eq. (4) as merely a shorthand notation for a complex summation. Indeed, on a different reading one can think that the relation of Eq. (4) should be treated as defining a local realistic observable itself, since it is of this quantity that Chen considers the variance and the expectation value of its square. However, to treat Eq. (4) as if it was defining an observable makes no difference at all for quantum mechanics, but as we have seen, for local realism this makes a crucial difference. A local realist would either encounter the problem mentioned in the first subtlety (and would then face the Bell-type critique), or the problem mentioned in the third subtlety (and would then not give any interesting results, i.e., local realism can then trivially reproduce the quantum predictions).

We thus see that on both readings Chen's analysis breaks down.

Acknowledgements

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[1] Z. Chen, ‘Variants of Bell inequalities’, quant-ph/0611126 (2006).

[2] M. Żukowski, ‘Separability of Quantum States vs. Original Bell (1964) Inequalities’, Found. Phys. **36**, 541 (2006).

[3] Note that in order to distinghuish between the local realistic observables (i.e., measurement functionals) and the quantum mechanical operators, I denote the quantum mechanical operators that correspond to observables in the local realistic expression by a ‘hat’. This difference will become crucial, which will become clear later on.

[4] Consider a bipartite scenario. For local realism the variance $\Delta(A(\lambda)B(\lambda))$ of the joint observable $A(\lambda)B(\lambda)$ is given by $\int(A(\lambda))^2(B(\lambda))^2\,p(\lambda)\,d\lambda - (\int A(\lambda)B(\lambda)\,p(\lambda)\,d\lambda)^2$. In quantum mechanics the variance $\Delta(\hat{A}\otimes\hat{B})$ of the joint observable $\hat{A}\otimes\hat{B}$ for separable states $\rho = \sum_j p_j\rho_j^1\otimes\rho_j^2$ is given by $\sum_j p_j\mathrm{Tr}^1[\rho_j^1\hat{A}^2]\mathrm{Tr}^2[\rho_j^2\hat{B}^2] - (\sum_j p_j\mathrm{Tr}^1[\rho_j^1\hat{A}]\mathrm{Tr}^2[\rho_j^2\hat{B}])^2$. This can be reproduced by the local realisic variance $\Delta(A(\lambda)B(\lambda))$ since we have that $|\mathrm{Tr}^1[\rho^1\hat{A}]| = |\langle\hat{A}\rangle_\rho| \leq \max A(\lambda)$ (and analogously for B).

[5] N.D. Mermin, ‘Extreme Quantum Entanglement in a superposition of Macroscopicaly Distinct States’, Phys. Rev. Lett. **65**, 1838 (1990); S.M. Roy, V. Singh, ‘Test of Signal Locality and Einstein-Bell-Locality for Multiparticle Systems’, Phys. Rev. Lett. **67**, 2761 (1991); M. Ardehali, ‘Bell inequalities with a magnitude that grows exponentially with the number of particles’, Phys. Rev. A **46**, 5375 (1992); A.V. Belinskiĭ, D.N. Klyshko, ‘Interference of Light and Bell's Theorem’, Phys. Usp. **36**, 653 (1993).

[6] See for example the paper by Chen [1] for the explicit form of this inequality and the choice of coefficients $c(k_1, k_2, \ldots, k_n)$ he uses. However, the explicit forms are not relevant here.

[7] R.F. Werner, M.M. Wolf, ‘All-multipartite Bell-correlation inequalities for two dichotomic observables per site’, Phys. Rev. A **64**, 032112 (2002).

[8] The operator $\hat{\mathcal{M}}_n$ has the spectral decomposition $\hat{\mathcal{M}}_n = 2^{(n-1)/2}(|\,\mathrm{GHZ}_+^n\rangle\langle\mathrm{GHZ}_+^n| - |\,\mathrm{GHZ}_-^n\rangle\langle\mathrm{GHZ}_-^n|)$, and the eigenvalues are (i) the values $\pm 2^{(n-1)/2}$ with n-partite GHZ-states $|\,\mathrm{GHZ}_\pm^n\rangle = 1/\sqrt{2}(|\,0^n\rangle \pm |\,1^n\rangle)$ as eigenstates and (ii) furthermore the value 0 with degeneracy $n-2$. The operator $\hat{\mathcal{M}}_n^2$ has the spectral decomposition $\hat{\mathcal{M}}_n^2 = 2^{(n-1)}(|\,0^n\rangle\langle 0^n| - |\,1^n\rangle\langle 1^n|)$, and eigenvalues are (i) the value $2^{(n-1)}$ (degeneracy 2) with separable eigenstates $|\,0^n\rangle$ and $|\,1^n\rangle$ and (ii) furthermore the value 0 with degeneracy $n-2$.

[9] J.S. Bell, ‘On the problem of hidden variables in quantum mechanics’, Rev. Mod. Phys. **38**, 447 (1966).

[10] K. Chen, S. Albeverio, S-M. Fei, ‘Two-setting Bell inequalities for many qubits’, Phys. Rev. A **74**, 050101 (2006); W. Laskowski, et al., ‘Tight Multipartite Bell's Inequalities Involving Many Measurement Settings’, Phys. Rev. Lett. **93**, 200401 (2004); Z. Chen, 'Bell-Klyshko Inequalities to Characterize Maximally Entangled States of n Qubits', Phys. Rev. Lett. **93**, 110403 (2004); X-H. Wu, H-S. Zong, 'Violation of local realism by a system with N spin-1/2 particles', Phys. Rev. A **68**, 032102 (2003); N. Gisin, H. Bechmann-Pasquinucci, ‘Bell inequality, Bell states and maximally entangled states for n qubits’, Phys. Lett. A **246**, 1 (1998).

[11] A remarkable exception is the work of M. Żukowski and Č. Brukner, ‘Bell's Theorem for General N-Qubit States’, Phys. Rev. Lett. **88**, 210401 (2002). In this work they derive the same inequalities as in [7] (by an independent derivation) but do not use Bell polynomials, or any such shorthand notation. They explicitly state the expecation values as was done in Eq. (1). The notation might perhaps not be so elegant, but from a conceptual point of view it is to be preferred since the subtleties and problems as discussed in this paper are explicitly avoided.